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Higher loops beyond the SU(2) sector

Joseph A. Minahan

Department of Theoretical Physics
Box 803, SE-751 08 Uppsala, Sweden, and
Center for Theoretical Physics, Massachusetts Institute of Technology
Cambridge, MA 02139, U.S.A.
E-mail: joseph.minahan@teorfys.uu.se

Abstract: We consider the case of coherent gauge invariant operators in the SU(3) and SO(4) sectors. We argue that in many cases, these sectors can be closed in the thermodynamic limit, even at higher loops. We then use a modification of the Bethe equations which is a natural generalization of a proposal put forward by Serban and Staudacher to make gauge theory predictions for the anomalous dimensions for a certain class of operators in each sector. We show that the predictions are consistent with semiclassical string predictions at two loops but in general fail to agree at three loops. Interestingly, in both cases there is one point in the configuration space where the gauge theory and string theory predictions agree. In the SU(3) case it corresponds to a circular string with R-charge assignment (J, J, J).

Keywords: 1/N Expansion, AdS-CFT and dS-CFT Correspondence.

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Contents

1. Introduction

There have many recent successes in comparing the anomalous dimensions of long coherent single trace gauge invariant operators in $\mathcal{N}=4$ SYM with the energies of semiclassical string states (see [1] for a comprehensive list). The key ingredients that make such comparisons possible are the semiclassical string picture [2] and integrability. At the one-loop level it was shown that the $\mathcal{N}=4$ SYM dilatation operator for the SO(6) sector can be mapped to an integrable spin-chain [3]. In [4, 5] this was extended to the full one-loop dilatation operator, unifying the results in [3] with earlier results from QCD [6, 7].

With integrability, the computation of the anomalous dimension is reduced to solving a set of Bethe equations. For long coherent operators, these equations become one or more integral equations. These integral equations were first solved for states that are dual to spinning and circular strings [8, 9], and it was shown that these solutions were consistent with the one-loop semiclassical predictions in [10]–[12]. Such strings and their dual operators live in an SU(2) subsector of the full SU(2, 2|4) algebra for the dilatation operator. It was subsequently shown that *all* one-loop anomalous dimensions are consistent with the semiclassical string predictions in this sector, either by comparing all solutions [13], or by showing that the string sigma model action reduces to the Heisenberg magnet effective action in the limit where the gauge coupling is taken to zero [14].

It is also possible to make higher loop comparisons. Once one goes beyond one-loop, integrability requires including all orders in the perturbative Yang-Mills expansion. Nevertheless, it was first shown that the dilatation operator is consistent with integrability at the two-loop level within a closed SU(2) sector of the full $SU(2,2|4)$ symmetry algebra [15]. These authors went further, assuming that integrability was also present at three loops. This allowed them to make a conjecture for the three loop correction to the anomalous dimension for the Konishi multiplet which has been recently verified by [16] where they applied an explicit calculation in [17] to $\mathcal{N}=4$ SYM. It was later shown by Beisert that integrability is consistent with three loops for the larger closed $SU(2|3)$ subsector [18].

With higher loop integrability, one can look for a set of Bethe equations. This was done first by Serban and Staudacher, where they argued that the Inozemtsev chain was consistent with the known dilatation operator up to three-loops in the SU(2) sector [19]. With these modified Bethe equations, Serban and Staudacher were able to compute the anomalous dimensions for the coherent operators dual to the circular and spinning string in the SU(2) sector and showed that it agreed with the predicted semiclassical string values at two loops but failed to match at three loops [19]. It was subsequently shown that all coherent SU(2) solutions match the string predictions at two-loops [13]. It was also shown that the effective action of the spin chain matches the sigma model action at two loops [20]. Another proposed set of Bethe equations has also been put forth, which further assumes that BMN scaling [21] is present at all orders in perturbation theory [22]. This matches the Inozemtsev chain to three loop order but diverges from it at 4 loops.

It is also possible to make one-loop comparisons outside of the SU(2) sector. This was first done in [8], where it was shown that a solution of the full SO(6) Bethe equations was consistent with the spectrum of a semiclassical pulsating string [23]. This analysis was carried out for more general scenarios in [24, 25], where a continuous set of SO(6) and SU(3) solutions were found and shown to agree with the string predictions at one-loop [10, 26, 24, 27]. It was also recently shown how to compare the effective action for the one-loop SU(3) chain with the string model, where agreement was found, essentially confirming that all coherent one-loop solutions will match with the string predictions [28, 29].

A natural question to ask is how the analysis for the coherenet operators in the full SO(6) or SU(3) sectors can be extended to higher loops. At first glance this would seem problematic, since these sectors are not closed above one loop. For example, in the SU(3) sector three different scalar fields can mix into two fermion fields, preserving spin, R-charge and the bare dimension [18]. In the SO(6) sector, the problem is even more acute; there can be mixing into the full $SU(2,2|4)$ set of fields.

However, in the semiclassical limit this mixing should be suppressed. The states in the semiclassical limit are coherent; the quantum states at the individual spin sites vary slowly over the length of the chain. The mixing outside of the SU(3) or SO(6) is essentially a quantum fluctuation that is suppressed by $1/L$. In the case of coherent SU(3) states, we will explicitly show this. In the case of SO(6) in the semiclassical limit, it is possible

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to find solutions that are further restricted to an SO(4) subgroup, which is normally not closed even at one loop. Again, however, one expects mixing outside of this subgroup to be suppressed by $1/L$.

If the mixing is suppressed, then it is natural to assume that the SO(6) or SU(3) Bethe equations are modified in a way analagous to the SU(2) equations. For the SO(6) solutions, we will show for a general class of solutions that the integral equations reduce to two independent SU(2) integral equations. Each SU(2) equation is then modified in the way proposed by Serban and Staudacher [19]. A similar modification is done for the SU(3) equations. In these cases we will show that the anomalous dimensions are consisistent with the dual string predictions up to two loop order but in general fail at three loops. In both cases there is one exceptional point where the three-loop predictions agree.

The three-loop failure is to be expected, based on past failures of $1/L$ corrections to the BMN limit [30, 31, 32], or for other semiclassical predictions [9, 19, 22]. However, in our case we have an adjustable parameter and we find that there is one value, aside from the BMN limit, where there is three-loop agreement.

The paper is organized as follows. In section 2 we show why coherent SU(3) operators have their mixing into the fermion sector suppressed by a factor of $1/L$. In section 3 we show how a class of SO(6) solutions reduce to SO(4) solutions. In section 4 we consider rational examples for these reduced SO(6) solutions, which includes duals to pulsating strings with an R-charge J. In section 5, we consider higher loop terms for the anomalous dimension, showing that the two loop prediction for pulsating string duals matches the semiclassical string prediction. We then show that the two loop SU(3) solution with R-charge assignment (J', J', J) is consistent with the string prediction. Next we compute the three loop terms for both cases, finding in general that the string and gauge theory predictions do not agree, except when $J = L/3$, where L is the bare dimension. In section 6 we compute the fluctuation spectra of the pulsating strings, using the fact that they are essentially reduced to the SO(4) sector. In section 7 we present our conclusions. An appendix contains some more complicated expressions.

2. Mixing suppression for coherent SU(3) states.

In this section we demonstrate that coherent operators made up of three chiral scalar fields have suppressed mixing to operators with fermions. As was already stated, the SU(3) sector is not closed under mixing, but is instead enlarged to the sector SU(2|3), where the mixing can first appear at the two-loop level [18]. To see this, we note that there is a two loop process where three scalar fields are converted to two fermion fields, as is shown in figure 1(a). There is also a one-loop process where two fermions convert to 3 scalars, as is shown in figure 1(b). Both processes come with a factor of g^3, but they also come with a factor of $N^{3/2}$. This is because the two loop term has a factor of N^2 but this is divided by a factor of $N^{1/2}$ since the operator with the extra three scalars has one more field than the operator with two fermions and each field comes with a normalization factor of $N^{-1/2}$. The overall effect of these mixing terms is a shift of the anomalous dimension by a term of λ^2.

Figure 1 a Figure 1 b

Figure 1: (a) Mixing of three scalars (dashed lines) into two fermions (solid lines). (b) Mixing of two fermions into three scalars.

Under the SU(3) $\times$ U(1) subgroup of SU(4), the chiral scalars transform in the $\mathbf{3_{+1}}$ representation, while the fermions in the $(\mathbf{2},\mathbf{1})$ representation of the Lorentz group transform in the $\mathbf{1_{3/2}}+\mathbf{3_{-1/2}}$ representation. Hence the antisymmetrized triplet of scalars can mix with the Lorentz singlet of two SU(3) singlet fermions. The contribution to the mixing matrix is a sum of local terms of the form [18]

$$H_3 = C\varepsilon_{\alpha\beta}\varepsilon^{abc}\left\{\begin{matrix}\alpha\beta\\ abc\end{matrix}\right\} + C^*\varepsilon_{abc}\varepsilon^{\alpha\beta}\left\{\begin{matrix}abc\\ \alpha\beta\end{matrix}\right\}, \tag{2.1}$$

where α and β are the SU(2) Lorentz indices and a, b, c are the SU(3) flavor indices. C is constant. The bottom line in the brackets refers to three (two) neigboring fields in the *in* state and the top line refers to two (three) neighboring fields in the *out* state. The effect of this term is to make the chain dynamical since it either increases or decreases the number of sites in the chain by one [18].

Now suppose we consider the SU(3) chain with nearest neighbor interactions in the classical limit. This was recently described in [28, 29]. The idea is to write the states in terms of collective coordinates. For an SU(3) transformation on a state in the fundamental representation, there is an SU(2) subgroup that leaves the state invariant, plus a U(1) that multiplies the state by a phase. Hence the collective coordinates are coordinates on the coset SU(3)/ SU(2) $\times$ U(1). These are described by 4 angles, with a general state written as [28, 29]

$$|\vec{n}(\sigma)\rangle = \cos\theta(\sigma)\cos\psi(\sigma)e^{i\varphi(\sigma)}|1\rangle + \cos\theta(\sigma)\sin\psi(\sigma)e^{-i\varphi(\sigma)}|2\rangle + \sin\theta(\sigma)e^{i\phi(\sigma)}|3\rangle\,, \tag{2.2}$$

where ϕ and φ range between 0 and 2π and θ and ψ range between 0 and $\pi/2$. σ labels the site on the chain, with neighboring sites differing by $\Delta\sigma = 2\pi/L$. The action can then be determined by considering the inner product $\langle\vec{n}(\sigma+\Delta\sigma)|\vec{n}(\sigma)\rangle$, from which one can determine the equations of motion [28, 29]

Consider then the action of H_3 in (2.1) on the state $|\vec{n}(\sigma-\Delta\sigma)\rangle|\vec{n}(\sigma)\rangle|\vec{n}(\sigma+\Delta\sigma)\rangle$ with the angles slowly varying between the sites. The resulting state has the form

$$H_3|\vec{n}(\sigma-\Delta\sigma)\rangle|\vec{n}(\sigma)\rangle|\vec{n}(\sigma+\Delta\sigma)\rangle = C(\Delta\sigma)^3 e^{i\phi(x)}F(\theta(x),\psi(x),\varphi(x),\phi(x))\varepsilon_{\alpha\beta}|\alpha\rangle|\beta\rangle\,, \tag{2.3}$$

where $F(\theta(x),\psi(x),\varphi(x),\phi(x))$ is a sum of three derivative terms and $|\alpha\rangle$ and $|\beta\rangle$ are the fermion states at neighboring sites. In the thermodynamic limit F is finite for H_3 acting on a coherent state.

At the one-loop level, an operator with chiral scalars and two fermion fields is closed; it only mixes with operators with two fermion fields. In fact the state with two fermions can be thought of as a fluctuation from a state with no fermions. There are of order L such fluctuations corresponding to the different choices of momentum for the fermions. These fluctuations increase the anomalous dimension by order $\lambda n^2/L^2$, with n an integer ranging from 1 to order L. The state with the fermions next to each other on the chain is approximately a linear combination of the momentum states with each coefficient $1/L$. But there are L sites on the chain for the transition to happen so the matrix element between a zero fermion state and a two fermion state is approximately

$$(gN)^{3/2}\langle 2|H_3|0\rangle \sim \left(\frac{\lambda}{L^2}\right)^{3/2}. \tag{2.4}$$

Hence we expect these fermion fluctuations to change the anomalous dimension by order

$$\sum_n \frac{(\lambda/L^2)^3}{n^2\lambda/L^2} \sim \frac{\lambda^2}{L^4}. \tag{2.5}$$

In the semiclassical limit, the two-loop contribution is of order λ^2/L^3, so the contribution from the fermion fluctuations is suppressed by a factor of $1/L$.

3. SO(6) reduction to SO(4)

In this section we show how a certain class of solutions to the SO(6) Bethe equations reduce to $\mathrm{SO}(4) \simeq \mathrm{SU}(2)\times\mathrm{SU}(2)$ Bethe equations. The arguments appearing here are essentially a generalization of those in [24]. The SO(4) symmetry can be easily understood from the string side, where it has been shown that the semiclassical string duals are restricted to an $R\times S_3$ subspace [23, 24, 13]. The SO(4) symmetry corresponds to the isometry group of S_3.

We will consider single trace operators $\mathcal{O}$ made up of scalar fields only. The operators are not holomorphic; they contain $\overline{X}$, $\overline{Y}$ and $\overline{Z}$ scalar fields inside the trace. We will assume that the operators are highest weights of SO(6) representations with bare dimension L and with R-charges $(0,0,J)$. In terms of SO(6) Dynkin indices, these representations are denoted by $[0,J,0]$.

The SO(6) Bethe equations are given by

$$\begin{aligned}
\left(\frac{u_{1,i}+i/2}{u_{1,i}-i/2}\right)^L &= \prod_{j\neq i}^{n_1}\frac{u_{1,i}-u_{1,j}+i}{u_{1,i}-u_{1,j}-i}\prod_j^{n_2}\frac{u_{1,i}-u_{2,j}-i/2}{u_{1,i}-u_{2,j}+i/2}\prod_j^{n_3}\frac{u_{1,i}-u_{3,j}-i/2}{u_{1,i}-u_{3,j}+i/2}\\
1 &= \prod_{j\neq i}^{n_2}\frac{u_{2,i}-u_{2,j}+i}{u_{2,i}-u_{2,j}-i}\prod_j^{n_1}\frac{u_{2,i}-u_{1,j}-i/2}{u_{2,i}-u_{1,j}+i/2}\\
1 &= \prod_{j\neq i}^{n_3}\frac{u_{3,i}-u_{3,j}+i}{u_{3,i}-u_{3,j}-i}\prod_j^{n_1}\frac{u_{3,i}-u_{1,j}-i/2}{u_{3,i}-u_{1,j}+i/2},
\end{aligned} \tag{3.1}$$

where n_1, n_2 and n_3 denote the number of Bethe roots associated with each simple root of SO(6). For this choice, the Dynkin indices of this representation are given by $[n_1 - 2n_2, L - 2n_1 + n_2 + n_3, n_1 - 2n_3]$. The anomalous dimension is only directly related to the u_1 roots and is given by

$$\gamma = \frac{\lambda}{8\pi^2} \sum_{j=1}^{n_1} \frac{1}{(u_{1,j})^2 + 1/4} . \tag{3.2}$$

There is also a momentum condition,

$$1 = \prod_{j=1}^{n_1} \frac{u_{1,j} + i/2}{u_{1,j} - i/2} , \tag{3.3}$$

which is a consequence of the cyclicity of the trace in $\mathcal{O}$. We now consider the thermodynamic limit, where $n_i \sim L$. We also assume a "half-filling" condition for the roots, where $n_2 = n_3 = n_1/2$ and we further assume that the distribution of the u_2 roots is the same as the distribution of the u_3 roots, which clearly is consistent with (3.1).

As was argued in [8, 24, 13], the roots will lie along cuts in the complex plane. We now assume that the u_1 roots are on multiple cuts, but the u_2 and u_3 roots are on a single cut. Taking logs on both sides of (3.1) and rescaling by $u = xL$, the Bethe equations reduce to the integral equations

$$\begin{aligned}
\frac{1}{x} - 2\pi n_i &= 2 \fint_{\mathcal{C}_i} dx' \frac{\sigma(x')}{x - x'} + 2 \sum_{j \neq i} \int_{\mathcal{C}_j} dx' \frac{\sigma(x')}{x - x'} - 2 \int_{\mathcal{C}'} dx' \frac{\rho(x')}{x - x'} \qquad x \in \mathcal{C}_i \\
0 &= 2 \fint_{\mathcal{C}'} dx' \frac{\rho(x')}{x - x'} - \sum_j \int_{\mathcal{C}_j} dx' \frac{\sigma(x')}{x - x'} \qquad x \in \mathcal{C}' ,
\end{aligned} \tag{3.4}$$

where $\mathcal{C}_i$ are the cuts for the rescaled u_1 roots, $\mathcal{C}'$ is the cut for the rescaled u_2 and u_3 roots, and n_i labels the log branch. Roots along the same cut are on the same log branch. The root densities satisfy normalization conditions

$$\sum_j \int_{\mathcal{C}_j} \sigma(x') dx' = 2 \int_{\mathcal{C}'} \rho(x') dx' = \frac{n_1}{L} = \frac{L - J}{L} \equiv \alpha , \tag{3.5}$$

and the anomalous dimension and momentum condition in (3.2) and (3.3) become

$$\gamma = \frac{\lambda}{8\pi^2} \sum_j \int_{\mathcal{C}_j} \frac{dx \sigma(x)}{x^2} , \tag{3.6}$$

and

$$2\pi m = \sum_j \int_{\mathcal{C}_j} \frac{dx \sigma(x)}{x} . \tag{3.7}$$

With the conditions in (3.5), we can Hilbert transform the second equation in (3.4) to get the relation for $\rho(x)$

$$\rho(x) = -\frac{1}{2\pi^2} \sqrt{(x-a)(x-b)} \fint_a^b \frac{dx'}{x - x'} \frac{1}{\sqrt{(x'-a)(x'-b)}} \sum_j \int_{\mathcal{C}_j} dx'' \frac{\sigma(x'')}{x' - x''} , \tag{3.8}$$

where a and b are the end points of the cut $\mathcal{C}'$. Since the existence of a Bethe root requires the presence of its complex conjugate, we must have $b = a^*$. If we invert the order of integration and integrate over x' by deforming the contour, we find

$$\rho(x) = -\frac{1}{2\pi i}\sum_j \int_{\mathcal{C}_j} dx'' \frac{\sigma(x'')}{x - x''}\frac{\sqrt{(x-a)(x-a^*)}}{\sqrt{(x''-a)(x''-a^*)}}\,. \tag{3.9}$$

If we now reinsert $\rho(x)$ in (3.8) into (3.5), then

$$\int_{\mathcal{C}'} dx'\rho(x') = \frac{1}{2}\sum_j \int_{\mathcal{C}_j} dx'\sigma(x') + \frac{1}{2}\sum_j \int_{\mathcal{C}_j} \frac{dx\sigma(x')(a + a^* - x')}{\sqrt{(x'-a)(x'-a^*)}} = \frac{\alpha}{2}\,. \tag{3.10}$$

Comparing (3.10) with (3.5) we see that $a \to \infty$ while at the same time, $\mathrm{Re}\, a/\mathrm{Im}\, a \to 0$ and so $\mathcal{C}'$ essentially cuts the complex plane in two. In this case $\frac{\sqrt{(x-a)(x-a^*)}}{\sqrt{(x'-a)(x'-a^*)}} \to \epsilon(x,x')$ where $\epsilon(x,x') = \pm 1$ with the $+$ $(-)$ sign if x and x' are on the same (opposite) sides of $\mathcal{C}'$. If we now take this limit on eq. (3.9), we obtain

$$\int_{\mathcal{C}'} \frac{dx'\rho(x')}{x - x'} = \sum_{j'} \int_{\mathcal{C}_{j'}} \frac{dx'\sigma(x')}{x - x'}, \tag{3.11}$$

where the sum over the index j' refers to those cuts that are on the *opposite* side of $\mathcal{C}'$ from x.

If we now examine (3.4), we see that the effect of the roots on $\mathcal{C}'$ is to screen the cuts on either side from each other. In other words, the system has degenerated to two independent sets of roots, each satisfying SU(2) spin 1/2 thermodynamic Bethe equations. The contribution to the anomalous dimension is the sum of the contribution from each SU(2) sector. The same is true for the momentum. In fact, we now see that the momentum condition in (3.7) does not apply to each SU(2) individually, but only to their combination. These arguments may also be applied to operators containing fields transforming in SU(2,2) representations, where in the thermodynamic limit the SU(2,2) can degenerate into SU(1,1) $\times$ SU(1,1) [35].

As with the case for a single SU(2), it is convenient to consider the resolvants

$$G_\pm(x) = \sum_{j_\pm} \int_{\mathcal{C}_{j_\pm}} \frac{dx'\sigma_\pm(x')}{x - x'}\,, \tag{3.12}$$

where the $+$ $(-)$ refers to the roots on the right (left) of $\mathcal{C}'$. The resolvents satisfy the equations

$$G_\pm(x + i0) + G_\pm(x - i0) = \frac{1}{x} - 2\pi n_{j_\pm} \qquad\qquad x \in \mathcal{C}_{j_\pm}\,, \tag{3.13}$$

where the $n_{j_-} < n_{j_+}$ for all j_- and j_+. In order to insure the cyclicity of the trace in $\mathcal{O}$, the resolvents must satisfy

$$G_+(0) + G_-(0) = -2\pi m\,, \tag{3.14}$$

where m is an integer. Likewise, from (3.2) the anomalous dimension is given by

$$\gamma = -\frac{\lambda}{8\pi^2 L}\left(G'_+(0) + G'_-(0)\right). \tag{3.15}$$

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Finally, equation (3.5) leads to the asymptotic condition

$$G_+(x) + G_-(x) \approx \frac{\alpha}{x}, \qquad x \to \infty\,. \tag{3.16}$$

4. Rational examples

The simplest situation to consider is when there is a single cut contributing to each resolvent. In this case $G_+(x)$ and $G_-(x)$ have an algebraic solution. Let us assume that

$$-G_\pm(0) = 2\pi s_\pm \tag{4.1}$$

and that the cuts are on the branches n_+ and n_-. In order to satisfy the trace condition in (3.14), we have $s_+ + s_- = m$.

The resolvents are now given by [13]

$$G_\pm(x) = \frac{1}{2x}\left(1 + \sqrt{(2\pi n_\pm x)^2 + 4\pi(2s_\pm - n_\pm) + 1}\right) - \pi n_\pm\,, \tag{4.2}$$

where the branch of the square root is chosen to cancel the pole at $x = 0$. The asymptotic behavior for $G_\pm(x)$ is

$$G_\pm(x) \approx \frac{s_\pm/n_\pm}{x}\,, \tag{4.3}$$

and so comparing with (3.16) we see that $\alpha = (s_+/n_+) + (s_-/n_-)$. Thus, in terms of α and m we have

$$s_\pm = -\frac{(\alpha n_\mp - m)n_\pm}{n_\pm - n_\mp} \tag{4.4}$$

Note that $s_\pm/n_\pm > 0$ in order for the states to be physical, but unlike the case of one SU(2), it is possible to have $s_+/n_+ > 1/2$ or $s_-/n_- > 1/2$, as long as $\alpha \leq 1$.

Using (3.15), we see that the anomalous dimension is

$$\gamma = \frac{\lambda}{2L}\frac{2\alpha(1-\alpha)n_+^2 n_-^2 - (\alpha n_+ n_- + m^2)(n_+^2 + n_-^2) + m(n_+^3 + n_-^3 - (1-2\alpha)n_+ n_-(n_+ + n_-))}{(n_+ - n_-)^2}. \tag{4.5}$$

This is a more general solution than that given in [24], where the solutions there correspond to $n_+ = -n_-$, $m = 0$. On the string side, these solutions were recently described in [27, 33].

An interesting application would be to compute the $1/L$ corrections along the lines of [34].

5. Higher loops

5.1 SO(6) rationals at two loops

The two-loop dilatation operator has yet to be determined in the full SO(6) sector. However, it is known in the SU(2) subsector, so it is natural to just use the SU(2) results, assuming again that in the semiclassical limit an SO(6) spin chain state can reduce to an SO(4) spin chain solution.

At two loops the SO(6) sector is not closed under dilatations. But as per our discussion in section 2, we will assume that these effects are subdominant in the semiclassical limit, and that the mixing outside the SO(6) sector is suppressed by factors of $1/L$.

The two-loop modification for an SU(2) chain leads to the modified integral equations for the resolvents [19, 13]

$$\frac{1}{x}+\frac{2T}{x^3}-2\pi n_j=G_\pm(x+i0)+G_\pm(x-i0) \qquad x\in\mathcal{C}_{j\pm}\,. \tag{5.1}$$

where $T=\frac{\lambda}{16\pi^2L^2}$. The momentum to two loop order is given by

$$2\pi s_\pm=-G'_\pm(0)-TG''_\pm(0) \tag{5.2}$$

and the anomalous dimension is

$$\gamma=-2TL\left(G'_+(0)+\frac{T}{2}G'''_+(0)+G'_-(0)+\frac{T}{2}G'''_-(0)\right). \tag{5.3}$$

In [13] the two loop result for rational solutions was explicitly computed. Borrowing those results, we find that the two-loop contribution to γ for the rational SO(6) solution of the last section is

$$\gamma_2=-\frac{\lambda^2}{8L^3}\left[s_+(n_+-s_+)(n_+^2-3s_+(n_+-s_+))+s_-(n_--s_-)(n_-^2-3s_-(n_--s_-))\right]. \tag{5.4}$$

In the case where $n_\pm=\pm n$, $s_\pm=\pm n\alpha/2$, (5.4) simplifies to

$$\gamma_2=-\frac{\lambda^2m^4}{64L^3}\alpha(2-\alpha)(4-3\alpha(2-\alpha))\,. \tag{5.5}$$

We now claim that the result in (5.5) is consistent with the semiclassical solution of a string pulsating on S_5 with its center of mass revolving around an equator with angular momentum J. The discussion follows closely that of [24] and previous work in [23]. Let us consider a circular pulsating string expanding and contracting on S_5, and with a center of mass that is moving on an S_3 subspace. We will assume that the string is fixed on the spatial coordinates in AdS_5, so the relevant metric is

$$ds^2=R^2\left(-dt^2+\sin^2\theta d\psi^2+d\theta^2+\cos^2\theta d\Omega_3^2\right), \tag{5.6}$$

where $d\Omega_3$ is the metric on the S_3 subspace and $R^2=2\pi\alpha'\sqrt{\lambda}$. The string is stretched only along the ψ coordinate and is wrapped n times. Fixing a gauge $t=\tau$, the Nambu-Goto action reduces to

$$S=-n\sqrt{\lambda}\int dt\sin\theta\,\sqrt{1-\dot\theta^2-\cos^2\theta g_{ij}\dot\phi^i\dot\phi^j}\,, \tag{5.7}$$

where g_{ij} is the metric on S_3 and ϕ^i refers to the coordinates on S_3. In [] it was shown that the hamiltonian is

$$H=\sqrt{\Pi_\theta^2+\frac{g^{ij}\Pi_i\Pi_j}{\cos^2\theta}+n^2\lambda\sin^2\theta}\,, \tag{5.8}$$

where Π_θ and Π_i are the canonical momentum for θ and the angles on S_3. The square of H has the form of a hamiltonian for a particle on S_5 with an angular dependent potential $n^2\lambda\sin^2\theta$. The semiclassical limit corresponds to large quantum numbers for the canonical momenta, so the potential may be considered as a perturbation. This string configuration has a total S_5 angular momentum L, which is the bare dimension for its gauge dual operator. On an S_3 subspace the angular momentum is J, which is the R-charge for the dual operator.

The unperturbed wave-functions are solutions to the Schroedinger equation

$$E^2\Psi(w) = -\frac{4}{w}\frac{d}{dw}w^2(1-w)\frac{d}{dw}\Psi(w) + \frac{J(J+2)}{w}\Psi(w)\,, \tag{5.9}$$

where $w = \cos^2\theta$. We will assume that J and L are even and define $j = J/2$ and $\ell = L/2$. The normalized S_5 wave functions are then given by

$$\Psi_{\ell,j}(w) = \frac{\sqrt{2(\ell+1)}}{(\ell-j)!}\frac{1}{w^{j+1}}\left(\frac{d}{dw}\right)^{\ell-j} w^{\ell+j}(1-w)^{\ell-j}\,. \tag{5.10}$$

In [] the first order correction to E^2 was shown to be

$$\int_0^1 wdw\Psi_{\ell,j}(w)n^2\lambda(1-w)\Psi_{\ell,j}(w) = n^2\lambda\frac{2(\ell+1)^2-(j+1)^2-j^2}{(2\ell+1)(2\ell+3)} \approx \frac{n^2\lambda(\ell^2-j^2)}{2\ell^2}\,. \tag{5.11}$$

For the second order correction, the matrix elements $\langle\Psi_{\ell',j'}|(1-w)|\Psi_{\ell,j}\rangle$, $\{\ell',j'\}\neq\{\ell,j\}$ satisfy

$$\begin{aligned}\langle\Psi_{\ell',j'}|(1-w)|\Psi_{\ell,j}\rangle &= \int_0^1 wdw\Psi_{\ell',j}(w)(1-w)\Psi_{\ell,j}(w)\,\delta_{j',j}\\ &= \frac{1}{2}\left(\frac{(\ell'+j+1)(\ell'-j)}{(2\ell'+1)\sqrt{\ell'(\ell'+1)}}\delta_{\ell',\ell+1} + \frac{(\ell+j+1)(\ell-j)}{(2\ell+1)\sqrt{\ell(\ell+1)}}\delta_{\ell'+1,\ell}\right)\delta_{j',j}\\ &\approx \left(\frac{\ell^2-j^2-j}{4\ell^2}(\delta_{\ell',\ell+1}+\delta_{\ell'+1,\ell})+\frac{j^2}{4\ell^3}\left(3\delta_{\ell',\ell+1}+\delta_{\ell'+1,\ell}\right)\right)\delta_{j',j}\,.\end{aligned} \tag{5.12}$$

Thus, up to second order in λ/L^2 and assuming large L, J and λ, E^2 is given by

$$E^2 = L^2 + n^2\lambda\frac{L^2-J^2}{2L^2} + (n^2\lambda)^2\frac{(L^2-J^2)(L^2-5J^2)}{32L^6} + O\left(\frac{\lambda^3}{L^4}\right). \tag{5.13}$$

Hence, again up to second order in λ/L^2, E is

$$E = L + n^2\lambda\frac{L^2-J^2}{4L^3} - (n^2\lambda)^2\frac{(L^2-J^2)(L^2+3J^2)}{64L^7} + O\left(\frac{\lambda^3}{L^5}\right). \tag{5.14}$$

Replacing $J = L(1-\alpha)$ and comparing (5.14) to (5.5), we find agreement.

5.2 The SU(3) chain at two loops

Operators $\mathcal{O}$ containing the three complex scalar fields but not their conjugates have a one-loop dilatation operator that maps to a hamiltonian for an integrable SU(3) chain [3]. The dilatation operator is known to three loops and has been shown to be consistent with integrability [18]. However, the SU(3) operators are not closed at higher loops, instead they mix under an SU(2|3) subgroup of SU(2,2|4).

However, based on our earlier arguments we will assume that mixing with operators containing fermion fields can be ignored in the semiclassical limit. We will only consider states with R-charge assignment (J', J', J) and bare dimension $L = J + 2J'$. This corresponds to having no u_3 roots and half as many u_2 roots as u_1 roots [24]. We then make the ansatz that the two loop modification for the Bethe equations changes the lhs of the first equation in (3.4) to the lhs of the modified SU(2) Bethe equations in [19]. The upshot of all this is that the first equation in (3.4) gets modified to

$$\frac{1}{x} + \frac{2T}{x^3} - 2\pi n_i = 2 \fint_{\mathcal{C}_i} dx' \frac{\sigma(x')}{x - x'} + 2 \sum_{j \neq i} \int_{\mathcal{C}_j} dx' \frac{\sigma(x')}{x - x'} - \int_{\mathcal{C}'} dx' \frac{\rho(x')}{x - x'} \qquad x \in \mathcal{C}_i \,. \quad (5.15)$$

Let us suppose that the u_1 roots are distributed symmetrically on two cuts, $\mathcal{C}_+$ and $\mathcal{C}_-$, on either side of $\mathcal{C}'$ which lies on the imaginary axis. The branch numbers are assumed to be $n_+ = -n_- = n$. Then the resolvent $W(x)$,

$$W(x) = \int_{\mathcal{C}_+} \frac{dx' \sigma(x')}{x - x'} \,, \tag{5.16}$$

satifies the Riemann-Hilbert equation [24]

$$W(x + i0) + W(x - i0) - W(-x) = \frac{1}{x} + \frac{2T}{x^3} - 2\pi n \qquad x \in \mathcal{C}_+ \,. \tag{5.17}$$

As in [24], we write $W(x) = W_r(x) + w(x)$, where

$$W_r(x) = \frac{1}{3x} + \frac{2T}{3x^3} - 2\pi n \,, \tag{5.18}$$

and $w(x)$ satisfies the homogeneous Riemann-Hilbert equation

$$w(x + i0) + w(x - i0) - w(-x) = 0 \qquad x \in \mathcal{C}_+ \,. \tag{5.19}$$

Then the function

$$r(x) = w^2(x) - w(x)w(-x) + w(-x)^2 \tag{5.20}$$

is even and regular across $\mathcal{C}_+$. In terms of the filling fraction $\alpha = 2J'/L$, $w(x)$ is asymptotically

$$\begin{aligned} w(x) &\approx (2\pi n) + \left(\frac{\alpha}{2} - \frac{1}{3}\right)\frac{1}{x} \qquad & x \to \infty \\ w(x) &\approx -\frac{1}{3x} - \frac{2T}{x^3} \qquad & x \to 0 \,. \end{aligned} \tag{5.21}$$

Then, in order to be consistent with (5.21), $r(x)$ is given by

$$r(x) = (2\pi n)^2 + \frac{1}{3x^2} + \frac{4T}{3x^4} + \frac{pT}{x^2} \,, \tag{5.22}$$

to linear order in T, where the coefficient p is to be determined.

One then has the equation

$$w^3(x) - r(x)w(x) = w^3(-x) - r(-x)w(-x) \equiv s(x) \,, \tag{5.23}$$

where $s(x)$ is an odd function that is also regular across the cut since $w(-x)$ is regular there. The asymptotic conditions for $w(x)$ then give

$$s(x) = \frac{2}{27x^3} + (2\pi n)^2 \left(\alpha - \frac{2}{3}\right) \frac{1}{x} + \frac{4T}{9x^5} + \frac{qT}{x^3} \tag{5.24}$$

to linear order in T, where q is to be determined.

It is convenient to use $g(x) = w(x) - w(-x)$, in which case $g(x)$ satisfies the equation

$$g^3(x) - r(x)g(x) + s(x) = 0\,. \tag{5.25}$$

and the full resolvant $G(x)$ is given by $G(x) = g(x) + W_r(x) - W_r(-x)$. Solving (5.25) as an expansion about $x = 0$, we find that up to linear order in T, $g(x)$ is given by

$$g(x) = -\frac{4T}{x^3} - \frac{2}{3x} - \alpha(2\pi n)^2 \left(1 - 4T(1-\alpha)(2\pi n)^2 + pT\right) x - \alpha(1-2\alpha)(2\pi n)^4 + \cdots\,. \tag{5.26}$$

If we substitute the corresponding expression for $w(x)$ into (5.20), we find that $p = 2\alpha \times (2\pi n)^2$ to leading order in T. It then follows that $q = \frac{4}{3}(2\pi n)^2(2\alpha - 1)$.

The anomalous dimension to two loop order is now found to be

$$\gamma = -2TL\left(G'(0) + \frac{T}{2}G'''(0)\right) = \frac{n^2\alpha\lambda}{2L}\left(1 - \frac{n^2\lambda}{4L^2}\right). \tag{5.27}$$

This agrees with the result of Frolov and Tseytlin in [10].

5.3 Three loops

In this subsection we consider the 3-loop contributions for the pulsating string and the SU(3) circular string. More details can be found in the appendix. As with the SU(2) examples, the gauge predictions are in general different from the string predictions. However, both types of operators have a continuous parameter that can be adjusted, and curiously there is agreement at one point in each example.

At three loops, using the SU(2) results of the Inozemtsev chain [19], the integral equations are modified to

$$\frac{1}{x} + \frac{2T}{x^3} + \frac{6T^2}{x^5} - 2\pi n_i = 2\,⨍_{\mathcal{C}_i} dx' \frac{\sigma(x')}{x - x'} + 2\sum_{j \neq i} \int_{\mathcal{C}_j} dx' \frac{\sigma(x')}{x - x'} - \nu \int_{\mathcal{C}'} dx' \frac{\rho(x')}{x - x'} \qquad x \in \mathcal{C}_i\,, \tag{5.28}$$

where $\nu = 2$ for the SO(6) chain and $\nu = 1$ for the SU(3) chain. The modifications to the momentum and the anomalous dimension to third order in λ/L^6 are

$$\begin{aligned} 2\pi s &= -G(0) - G''(0)T - \frac{1}{4}G^{(4)}(0)T^2 \\ \gamma &= -2LT\left(G'(0) + \frac{1}{2}G'''(0) + \frac{1}{12}G^{(5)}(0)\right). \end{aligned} \tag{5.29}$$

In the case of SO(6) we again assume that the solution is reduceable to SO(4) $\simeq$ SU(2) $\times$ SU(2), with each SU(2) having one cut of roots. Hence we have that $G(x) = G_+(x) + G_-(x)$, where $G_\pm(x)$ is a single cut solution. For such a rational solution the contribution to the three-loop term is $\gamma^{(3)} = \gamma_+^{(3)} + \gamma_-^{(3)}$ where using (5.29) and results in the appendix

$$\gamma_\pm^{(3)} = \frac{\lambda^3 s_\pm(n_\pm - s_\pm)(n_\pm^2 - 3s_\pm(n_\pm - s_\pm))(2s_\pm - n_\pm)^2}{16L^5} + \mathrm{O}(\lambda^4/L^7) \tag{5.30}$$

If we now set $s_\pm = \pm\alpha n/2$ and $n_\pm = \pm n$, then

$$\gamma^{(3)} = \frac{\lambda^3 n^6 \alpha(2-\alpha)(4-3\alpha(2-\alpha))(1-\alpha)^2}{128L^5}\,. \tag{5.31}$$

The equation in (5.31) can be written more compactly in terms of $\kappa = (1-\alpha)^2$,

$$\gamma = \frac{\lambda n^2(1-\kappa)}{4L} - \frac{\lambda^2 n^4(1-\kappa)(1+3\kappa))}{64L^3} + \frac{\lambda^3 n^6(1-\kappa)(1+3\kappa)\kappa}{128L^5} + \mathrm{O}(\lambda^4/L^7)\,, \tag{5.32}$$

where we have included the one and two-loop terms for completeness. In the appendix, the contribution to the energy at order λ^3/L^5 is computed for the semiclassical string by doing third order perturbation theory. In terms of κ, this is

$$E^{(3)} = \frac{\lambda^3 n^6(1-\kappa)(1-3\kappa)(1-5\kappa)}{256L^5}\,, \tag{5.33}$$

and so does not match the corresponding term in (5.32). The difference between the terms is

$$E^{(3)} - \gamma^{(3)} = \frac{\lambda^3 n^6(1-\kappa)(1-\kappa)(1-9\kappa)}{256L^5}\,. \tag{5.34}$$

The terms match at $\kappa = 1$ as expected, since this is the BMN limit. But curiously they also match at $\kappa = 1/9$ which corresponds to $\alpha = 2/3$ and so $J = L/3$.

In the (J', J', J) SU(3) case the equation for the resolvent $W(x)$ in (5.17) is modified to

$$W(x+i0) + W(x-i0) + W(-x) = \frac{1}{x} + \frac{2T}{x^3} + 6T^2x^5 - 2\pi n \qquad x \in \mathcal{C}_+\,, \tag{5.35}$$

and so $W_r(x)$ is

$$W_r(x) = -\frac{1}{3x} - \frac{2T}{3x^3} - \frac{2T^2}{x^5} - 2\pi n\,. \tag{5.36}$$

Equations (5.19), (5.20) and (5.25) still apply, although $r(x)$ and $s(x)$ need to be adjusted since the asymptotic behavior of $w(x)$ is modified to

$$w(x) \approx -\frac{1}{3x} - \frac{2T}{x^3} - \frac{2T^2}{x^5} \qquad x \to 0\,. \tag{5.37}$$

The functions $r(x)$, $s(x)$ and the corresponding solution $g(x)$ up to order T^2 are given in the appendix. The resulting anomalous dimension is

$$\gamma = -2TL\left(G'(0) + \frac{T}{2}G'''(0) + \frac{T^2}{12}G^{(5)}(0)\right) = \frac{n^2\alpha\lambda}{2L}\left(1 - \frac{n^2\lambda}{4L^2} + \frac{n^4\lambda^2(1-\alpha^2)}{8L^4}\right)\,. \tag{5.38}$$

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Notice that there is no three loop contribution if $\alpha = 1$. This corresponds to $J = 0$ where the SU(3) chain reduces to the SU(2) chain, whose null three loop term was previously computed [19].

Taking the expansion for the Frolov-Tsytlin semiclassical string one loop higher, we find [10]

$$E = L + \frac{n^2\alpha\lambda}{2L}\left(1 - \frac{n^2\lambda}{4L^2} + \frac{n^4\lambda^2(1-2\alpha+2\alpha^2)}{8L^4}\right) + \mathrm{O}\left(\frac{\lambda}{L^7}\right). \qquad (5.39)$$

As is now becoming quite familiar, the third order terms in (5.38) and (5.39) do not match. But once again, they do match for the special case of $\alpha = 2/3$ ($J = L/3$), which in the circular string corresponds to all three R-charges being equal. In terms of the SU(3) subgroup of the SU(4) R-symmetry group, this state is an SU(3) singlet. As in the pulsating string case, it is not clear if this a happy coincidence or a clue toward resolving the divergence of the gauge and string expansions.

6. SO(6) Fluctuations

This section is somewhat outside the main development of this paper. For rational SU(2) solutions it is quite easy to find the spectrum of fluctuations about the one-loop semiclassical solution. We first present a method applicable to SU(2) solutions and then apply it to the SO(6) case.

The idea is to remove one Bethe root from the cut, essentially placing it on a different branch. Assuming a one cut Bethe solution with momentum $2\pi s$ and branch number n, then if one root is moved onto a branch with branch number $n_1 = n + \Delta n$, then the position of the root x_1 in the rescaled complex plane satisfies the equation

$$\frac{1}{x_1} - 2\pi n_1 = 2\int_{\mathcal{C}} \frac{dx'\sigma(x')}{x_1 - x'} = 2G(x_1)\,, \qquad (6.1)$$

where $G(x)$ has the form in (4.2). Hence x_1 is given by

$$\begin{aligned} x_1 &= \frac{1}{2\pi}\frac{(2s-n) \pm \sqrt{(\Delta n)^2 - 4s(n-s)}}{(\Delta n)^2 - n^2} & \Delta n &\neq n \\ x_1 &= \frac{1}{4\pi(n-2s)} & \Delta n &= n\,. \end{aligned} \qquad (6.2)$$

To compute the change in the anomalous dimension due to this movement of the root, we need to compute the back reaction on the cut [8, 36]. The effect of this root is modify the integral equation for the cut to

$$\frac{1}{x} - 2\pi n = 2 \,⨍_{\mathcal{C}} \frac{dx'\sigma(x')}{x - x'} + \frac{2/L}{x - x_1} \qquad x \in \mathcal{C}\,, \qquad (6.3)$$

where x_1 is given in (6.2). Hence, we need to solve the Riemann-Hilbert problem

$$G(x + i0) + G(x - i0) = \frac{1}{x} - 2\pi n - \frac{2/L}{x - x_1}\,. \qquad (6.4)$$

The general form for $G(x)$ is

$$G(x) = \frac{1}{2x} - \frac{1/L}{x - x_1} + \frac{1}{2}\left(\frac{1}{x} + \frac{b}{x - x_1}\right)\sqrt{A^2x^2 + Bx + 1} - \pi n\,. \tag{6.5}$$

In order to cancel the pole at $x = x_1$, we have that

$$\frac{b}{2}\sqrt{A^2x^2 + Bx + 1} = \frac{1}{L}\,. \tag{6.6}$$

Approximating A and B by their values in (4.2), we obtain

$$b \approx -\frac{1}{\pi L \Delta n x_1}\,. \tag{6.7}$$

In order to have the correct asymptotic behavior for $G(x)$, we also have

$$\begin{aligned} A &\approx 2\pi n(1 - b) \\ B &\approx 4\pi(2s - n)(1 - 2b) + \frac{8\pi n^2}{L\Delta n}\,. \end{aligned} \tag{6.8}$$

The momentum is given by

$$\frac{1}{x_1} - G(0) = 2\pi s + \frac{2\pi\Delta n}{L} \tag{6.9}$$

and the one-loop anomalous dimension is

$$\gamma = \frac{\lambda}{8\pi^2}\left(\frac{1}{L^2x_1^2} - \frac{G'(0)}{L}\right). \tag{6.10}$$

Hence the change in the anomalous dimension due to the fluctuation is

$$\Delta\gamma = \frac{\lambda\Delta n}{2L^2}\left(\sqrt{(\Delta n)^2 - 4s(n - s)} - 2(2s - n)\right). \tag{6.11}$$

This matches the form in [10, 27].

Examining (6.11) we see that if $(\Delta n)^2 < 4s(n - s)$ then $\Delta\gamma$ is complex, signaling an instability. If we look at (6.2), we see that this corresponds to x_1 moving off the real axis, an effect also seen in [36]. A physical Bethe state must have Bethe roots that are real, or are in conjugate pairs. Thus the instability for the quantum state can be thought of as a Bethe root pushed off into the forbidden region. However, an allowed state can still have such a root if it also has the conjugate.

For circular strings, one has that $s = m > 0$ and that $n \geq 2m$. Hence circular strings always have an instability for the $\Delta n = \pm 1$ mode since $s(n - s) \geq 1$.

For pulsating strings, if we have $s_+ = -s_-$ and $n_+ = -n_- = 1$, then there is no instability. The string with $\alpha = 1$ is just barely stable, since here one finds that the $\Delta n = 1$ mode is massless. Note that the physical states still have to satisfy the momentum condition. Hence a state requires at least two fluctuations, with the sum over all Δn being zero.

7. Discussion

In this paper we have exploited the "closed sector reduction" for certain types of long coherent operators in order to compute higher loop contributions to their anomalous dimensions. At the two-loop level we find agreement with string predictions for the anomalous dimensions. It should be straightforward to verify that all higher charges match as well, using arguments and procedures given in [37, 38, 13, 39]. It would also be interesting to consider the higher loop contributions for more general operators in the SU(3) sector, for example, those considered in [25].

At three-loops there is disagreement except for the special cases. The Inozemtsev model does not have perturbative BMN scaling at four loops [19] and so there can be no way to match the gauge and string theory predictions at the four-loop level with this model. However, very recently a new model was proposed for the SU(2) chain which assumes that BMN scaling holds to all orders in perturbation theory [22]. One could modify the Bethe equations for the SU(3) and SO(6) chains as outlined here. We would not expect agreement for general operators at four loops or higher, but it would be interesting to see if the (J, J, J) state agrees at this level.

Resolving the mystery of three loops is a crucial problem in our understanding of the AdS/CFT correspondence [40]–[42]. If the correspondence is correct, then presumably this is an issue of strong versus weak coupling and there are contributions that do not appear in perturbation theory but do contribute to the classical string, or vice versa. In the examples provided here, while there is disagreement at three loops, the difference between the predictions are simple rational expressions. Quantities that are rational are often computable, giving one hope that ultimately a resolution can be found.

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A. Some results for three loops

A.1 Single cut SU(2)

At three loops, the Riemann Hilbert problem becomes

$$G(x+i0)+G(x-i0)=\frac{1}{x}+\frac{2T}{x^3}+\frac{6T^2}{x^5}-2\pi n \qquad x\in\mathcal{C}\,. \tag{A.1}$$

With only one cut on branch n, the general form of $G(x)$ is

$$G(x)=\frac{1}{2x}+\frac{T}{x^3}+\frac{3T^2}{x^5}+\frac{1}{2}\left(\frac{a_1}{x}+\frac{a_2}{x^2}+\frac{a_3}{x^3}+\frac{a_4}{x^4}+\frac{3T^2}{x^5}\right)\sqrt{A^2x^2+Bx+1}-\pi n\,. \tag{A.2}$$

We also set the total momentum to $2\pi s$ and so $G(x)$ is asymptotically

$$G(x)\approx\frac{s/n}{x} \qquad x\to 0\,, \tag{A.3}$$

since the number of roots is s/n. The a_i can be determined in terms of A and B by canceling the singularities at $x = 0$. The asymptotic conditions then give us the two further equations

$$\begin{aligned} a_1 A &= 2\pi n \\ 1 + \frac{a_1 B}{2A} + a_2 A &= \frac{s}{n}\,. \end{aligned} \tag{A.4}$$

The results are

$$\begin{aligned} a_1 &= 1 - \left(A^2 - \frac{3}{4}B^2\right)T + \left(\frac{9}{4}A^4 - \frac{45}{8}B^2A^2 + \frac{105}{64}B^4\right)T^2 \\ a_2 &= -BT + \left(\frac{9}{2}BA^2 - \frac{15}{8}B^3\right)T^2 \\ a_3 &= 2T - \left(A^2 - \frac{9}{4}B^2\right)T^2 \\ a_4 &= -3BT^2\,. \end{aligned} \tag{A.5}$$

These equations and the conditions in (A.4) lead to the following approximations for the variables:

$$\begin{aligned} A &= 2\pi n\Big(1 - 2T(2\pi)^2(n^2 - 6s(n-s)) + && \text{(A.6)} \\ &\qquad + 6T^2(2\pi)^4(n^4 - 16n^3s + 66n^2s^2 - 100n^s + 50s^4)\Big) + \mathrm{O}(T^3) \\ B &= 4\pi(2s-n)\Big(1 - 2T(2\pi)^2(n^2 - 12s(n-s)) + && \text{(A.7)} \\ &\qquad + 6T^2(2\pi)^4(n^4 - 30n^3s + 154n^2s^2 - 248ns^3 + 124s^4)\Big) + \mathrm{O}(T^3) && \text{(A.8)} \\ a_1 &= 1 + 2T(2\pi)^2(n^2 - 6s(n-s)) - 2T^2(2\pi)^4(n^4 - 24n^3s + 102n^2s^2 - 156ns^3 + 78s^4) \\ &\quad + \mathrm{O}(T^3) \\ a_2 &= -2T(2\pi)(2s-n) - 2T^2(2\pi)^3(2s-n)(n^2 - 6s(n-s)) + \mathrm{O}(T^3) \\ a_3 &= 2T + 6T^2(2\pi)^2(n^2 - 6s(n-s)) + \mathrm{O}(T^3) \\ a_4 &= -6T^2(2\pi)(2s-n) + \mathrm{O}(T^3)\,. && \text{(A.9)} \end{aligned}$$

Expanding $G(x)$ about $x = 0$ we find

$$\begin{aligned} G'(0) &= -(2\pi)^2 s(n-s) - 2T(2\pi)^4 s(n-s)(n-3s)(2n-3s) \\ &\quad - 2T^2(2\pi)^6 s(n-s)(8n^2 - 3s(n-s)(29n^2 - 74s(n-s))) + \mathrm{O}(T^3) \\ \frac{1}{3!}G'''(0) &= -(2\pi)^4 s(n-s)(n^2 - 5s(n-s)) \\ &\qquad + 4T(2\pi)^6 s(n-s)(2n^2 - 5s(n-s)(5n^2 - 14s(n-s))) + \mathrm{O}(T^2) \\ \frac{1}{5!}G^{(5)}(0) &= -(2\pi)^6 s(n-s)(n^4 - 14s(n-s)(n^2 - 3s(n-s))) + \mathrm{O}(\mathrm{T})\,. \end{aligned} \tag{A.10}$$

A.2 Terms for SU(3)

The functions $r(x)$ and $s(x)$ are determined by matching the singularities in equations (5.20), (5.21), (5.37) and (5.25). The results are

$$\begin{aligned} r(x) &= \frac{1}{3x^2} + (2\pi n)^2 + \left(\frac{4}{3x^4} + \frac{2(2\pi n)^2\alpha}{x^2}\right)T + \left(\frac{16}{3x^6} + \frac{6(2\pi n)^2\alpha}{x^4} - \frac{2(2\pi n)^4\alpha}{x^2}\right)T^2 + \\ &\quad + \mathrm{O}(T^3)\,, \\ s(x) &= \frac{2}{27x^3} - \frac{(2\pi n)^2(2-3\alpha)}{3x} + \left(\frac{4}{9x^5} - \frac{4(2\pi n)^2(1-2\alpha)}{3x^3}\right)T \\ &\quad + \left(\frac{20}{9x^7} - \frac{4(2\pi n)^2(3-7\alpha)}{3x^5} + \frac{2(2\pi n)^4\alpha(2-3\alpha)}{3x^3}\right)T^2 + \mathrm{O}(T^3)\,. \end{aligned} \tag{A.11}$$

We then find for $g(x)$

$$\begin{aligned} g(x) &= \left(-\frac{2}{3x} - \alpha(2\pi n)^2 x - \alpha(1-2\alpha)(2\pi n)^4 x^3 - \alpha(1-6\alpha+7\alpha^2)(2\pi n)^6 x^5 + \mathrm{O}(\mathrm{x}^7)\ldots\right) + \\ &\quad + \left(-\frac{4}{3x^3} + 2\alpha(2-3\alpha)(2\pi n)^4 x + 8\alpha(1-5\alpha(1-\alpha))(2\pi n)^6 x^3 + \mathrm{O}(x^5)\right)T + \\ &\quad + \left(-\frac{4}{x^5} - 4\alpha(4-15\alpha+12\alpha^2)(2\pi n)^6 x + \mathrm{O}(x^3)\right)T^2 + \mathrm{O}(T^3)\,. \end{aligned} \tag{A.12}$$

A.3 Third order perturbation theory

The general expression for the third order correction to the energy from a perturbation $\mathcal{H}'$ to a hamiltonian is

$$\varepsilon^{(3)} = \sum_{\ell'\neq\ell,\ell''\neq\ell} \frac{\langle\ell|\mathcal{H}'|\ell'\rangle\langle\ell'|\mathcal{H}'|\ell''\rangle\langle\ell''|\mathcal{H}'|\ell\rangle}{(\varepsilon_\ell-\varepsilon_{\ell'})(\varepsilon_\ell-\varepsilon_{\ell''})} - \sum_{\ell'\neq\ell} \frac{\langle\ell|\mathcal{H}'|\ell'\rangle\langle\ell'|\mathcal{H}'|\ell\rangle\langle|\ell|\mathcal{H}'|\ell\rangle}{(\varepsilon_\ell-\varepsilon_{\ell'})^2}\,. \tag{A.13}$$

In our case $\mathcal{H}' = \lambda n^2(1-w)$ and so $\langle\ell'|\mathcal{H}'|\ell\rangle \neq 0$ only if $\ell' = \ell\pm1$, $\ell'=\ell$. Hence, the expression in (A.13) reduces to

$$\varepsilon_\ell^{(3)} = \sum_{\ell'=\ell\pm1} \frac{|\langle\ell|\mathcal{H}'|\ell'\rangle|^2}{(\varepsilon_\ell-\varepsilon_{\ell'})^2}(\langle\ell'|\mathcal{H}'|\ell'\rangle - \langle\ell|\mathcal{H}'|\ell\rangle)\,. \tag{A.14}$$

If we define $\phi(\ell+1/2) = \langle\ell|\mathcal{H}'|\ell+1\rangle$ and $\psi(\ell) = \langle\ell|\mathcal{H}'|\ell\rangle$, then for large ℓ, we can approximate $\varepsilon_\ell^{(3)}$ by

$$\varepsilon_\ell^{(3)} = \frac{\phi^2(\ell)\psi''(\ell)}{(\varepsilon'(\ell))^2} + 2\frac{\phi(\ell)\phi'(\ell)\psi'(\ell)}{(\varepsilon'(\ell))^2} - 2\frac{\varepsilon''(\ell)\phi^2(\ell)\psi'(\ell)}{(\varepsilon'(\ell))^3}\,, \tag{A.15}$$

where from (5.11) and (5.12), we can approximate

$$\psi(\ell) = 2\phi(\ell) = \lambda n^2\frac{\ell^2-j^2}{2\ell^2} \qquad \varepsilon(\ell) = \ell^2\,. \tag{A.16}$$

Hence we find that

$$\varepsilon_\ell^{(3)} = (\lambda n^2)^3\frac{\kappa(1-\kappa)(9\alpha-5)}{64L^4}\,, \tag{A.17}$$

where $\kappa = j^2/\ell^2$. Hence, up to three loops, the energy squared of the string state is

$$E^2 = L^2 + \frac{\lambda n^2(1-\kappa)}{2} + \frac{(\lambda n^2)^2(1-\kappa)(1-5\kappa)}{32L^2} + \frac{(\lambda n^2)^3\kappa(1-\kappa)(9\kappa-5)}{64L^4} + \mathrm{O}(\lambda^4/L^6) \tag{A.18}$$

and so E is

$$E = L + \frac{\lambda n^2(1-\kappa)}{4L} - \frac{(\lambda n^2)^2(1-\kappa)(1+3\kappa)}{64L^3} + \frac{(\lambda n^2)^3(1-\kappa)(1-3\kappa)(1-5\kappa)}{256L^5}\,. \tag{A.19}$$

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